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Fixing the conformal window in QCD*

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Abstract: A physical characterization of Landau singularities is emphasized, which should trace the lower boundary N_f^* of the conformal window in QCD and supersymmetric QCD. A natural way to disentangle "perturbative" from "non-perturbative" contributions to amplitudes below N_f^* is suggested. Assuming an infrared fixed point persists in the perturbative part of the QCD coupling even below N_f^* leads to the condition $\gamma(N_f^*) = 1$, where γ is the critical exponent. Using the Banks-Zaks expansion, one gets $4 \leq N_f^* \leq 6$. This result is incompatible with the existence of an analogue of Seiberg duality in QCD. The presence of a negative ultraviolet fixed point is required both in QCD and in supersymmetric QCD to preserve causality within the conformal window.

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Contents

1. Introduction

The notion of an infrared (IR) finite coupling has been used [1] extensively in recent years, in particular in connection with the phenomenology of power corrections in QCD. The present investigation is motivated by the desire to understand better the theoretical background behind such an assumption. In particular, given an IR finite coupling α, does it remain finite within perturbation theory itself (such as the two-loop coupling with opposite signs one and two loop beta function coefficients), or does one need a non-perturbative contribution $\delta\alpha$ to cancell the Landau singularities present in its perturbative part α_{PT}? The answer I shall suggest is a mixed one: the perturbative part of the QCD coupling is *always* IR finite but, below the so called "conformal window" (the range of N_f values where the theory is scale invariant at large distances and flows to a non-trivial IR fixed point), one still needs a $\delta\alpha$ term since the perturbative coupling is no more causal there, despite being IR finite. As the main outcome, one obtains an equation to determine the lower boundary of the conformal window in QCD. The plan of the talk (a continuation of a similar one given at the "Gribov 70" meeting) is as follows. In section 2 I review the evidence and present a formal argument for the existence of Landau singularities in the perturbative coupling. A more physical argument, relating Landau singularities to the very existence of the conformal window and a two-phase structure of QCD is given in section 3, which also suggests a clean way to disentangle "perturbative" from "non-perturbative" below the conformal window. In section 4, two scenarios for causality breaking are described. In section 5, an equation to determine the bottom of the conformal window in QCD is suggested, and is solved through the Banks-Zaks expansion in section 6. Section 7 contains the conclusions.

2. Evidence for Landau singularities in the perturbative coupling

The only present evidence for a Landau singularity in the perturbative *renormalized*[1] coupling is still the old Landau -Pomeranchuk leading log QED calculation, now reformulated in QCD as a $N_f \to -\infty$ ("large β_0") limit. In this limit, the perturbative

[1] In QED, the well established "triviality" property gives only direct evidence [2] for a singularity in the *bare* coupling constant.

coupling is one-loop:

$$\alpha_{PT}(k^2) = \frac{1}{\beta_0 \log \frac{k^2}{\Lambda^2}} \tag{2.1}$$

where Λ is the Landau pole. The question is whether there is a singularity at *finite* N_f. Some light on this problem can be shed by considering the N_f dependence. Indeed, another (conflicting) piece of information is available at the other end of the spectrum, around the value $N_f = N_f^0 = 16.5$ (I consider $N_c = 3$) where the one loop coefficient β_0 of the beta function vanishes ("small β_0" limit). For N_f slightly below 16.5 a weak coupling (Banks-Zaks) IR fixed point develops [3, 4, 5] ($\alpha_{IR} \simeq -\frac{\beta_1}{\beta_0}$), and the perturbative coupling is causal, i.e. there are no Landau singularities in the whole first sheet of the complex k^2 plane. Can then the perturbative coupling remain causal down to $N_f = -\infty$? I shall assume that a Landau singularity cannot arise "spontaneously" in the limit, i.e. that "the limit of a sequence of causal couplings must itself be causal". In such a case, the existence of a Landau pole at $N_f \to -\infty$ implies the existence of a *finite* value N_f^* of N_f below which Landau singularities appear on the first sheet of the complex k^2 plane and perturbative causality is lost, which is the common wisdom (at N_f^* itself, according to the above philosophy, the coupling must still be causal). The range $N_f^* < N_f < N_f^0$ where the *perturbative* coupling is causal and flows to a finite IR fixed point is taken as the definition of the "conformal window" for the sake of the present discussion (this definition will be refined in the next section). I shall propose in section 5 an ansatz to determine N_f^* (the bottom of the conformal window) in QCD, but first let us give a more physical argument in favor of the existence of Landau singularities, which also illuminates their physical meaning.

3. Landau singularities and conformal window

Let us assume the existence of a two-phase structure in QCD as the number of flavors N_f is varied.

i) For $N_f^* < N_f < N_f^0$ (the conformal window) the theory is scale invariant at large distances, and the vacuum is "perturbative", in the sense there is no confinement nor chiral symmetry breaking. Conformal window amplitudes (generically noted as $D_{\overline{PT}}(Q^2)$, where Q stands for an external scale) are in this generalized sense "perturbative", i.e. could in principle be determined from information contained in perturbation theory to all orders (although they should also include contributions from all instanton sectors): this motivates the subscript $\overline{PT}$.

ii) For $0 < N_f < N_f^*$ there is a phase transition to a non-trivial vacuum, with confinement and chiral symmetry breaking, as expected in standard QCD.

A direct, *physical* motivation for Landau singularities can now be given: they trace the lower boundary $N_f = N_f^*$ of the conformal window. This statement is implied from the following two postulates:

1) Conformal window amplitudes $D_{\overline{PT}}(Q^2)$ can be analytically continued in N_f below the bottom N_f^* of the conformal window.

2) For $N_f < N_f^*$, the (analytically continued) conformal window amplitudes $D_{\overline{PT}}(Q^2)$ must *differ* from the *full* QCD amplitude $D(Q^2)$, since one enters a new phase, i.e. we have

$$D(Q^2) = D_{\overline{PT}}(Q^2) + D_{\overline{NP}}(Q^2) \tag{3.1}$$

(whereas $D(Q^2) \equiv D_{\overline{PT}}(Q^2)$ within the conformal window). Assuming QCD to be a *unique* theory at given N_f, $D_{\overline{PT}}(Q^2)$ cannot provide a consistent solution if $N_f < N_f^*$: this must be signalled by the appearance of unphysical Landau singularities in $D_{\overline{PT}}(Q^2)$ for $N_f < N_f^*$. N_f^* should thus coincide with the value of N_f below which Landau singularities first appear in $D_{\overline{PT}}(Q^2)$. The occurence of a "genuine" non-perturbative component $D_{\overline{NP}}(Q^2)$ is then necessary below N_f^* in order to cancell the Landau singularities present in $D_{\overline{PT}}(Q^2)$. If these assumptions are correct, they provide an interesting connection between information contained in principle in "perturbation theory" (over all instanton sectors), which fix the structure of the conformal window amplitudes and "genuine" non-perturbative phenomena, which fix the bottom of the conformal window. In addition, eq.(3.1) provide a neat way to disentangle the "perturbative" from the genuine "non-perturbative" part of an amplitude, for instance the part of the gluon condensate related to renormalons from the one reflecting the presence of the non-trivial vacuum. Note also $D_{\overline{PT}}(Q^2)$ and $D_{\overline{NP}}(Q^2)$ are separately free of renormalons ambiguities, but contain Landau singularities below N_f^*, so the renormalon and Landau singularity problems are also disentangled. In order to get a precise condition to determine N_f^*, we need now to look in more details how causality can be broken in the perturbative coupling.

4. Scenarios for causality breaking

One can distinguish two main scenarios:

i) The "standard" scenario where the IR fixed point present within the conformal window just disappears when $N_f < N_f^*$ and a real, space-like Landau singularity is generated in the perturbative coupling. For instance, two simple zeroes of the beta function can merge into a double zero when $N_f = N_f^*$ before moving to the complex plane (a plausible scenario [6] in supersymmetric QCD (SQCD)).

ii) Alternatively, it is possible for the fixed point *to be still present*[2] in the *perturbative part* of the coupling for $N_f < N_f^*$. The motivation behind this assumption is the observation [7, 8, 6] that, at least for some quantities, the Banks-Zaks expansion in QCD (as opposed to SQCD [6]) seems to converge down to fairly small values of

[2]This assumption is consistent with the suggestion of [7] that the *perturbative* coupling has a non-trivial IR fixed point down to $N_f = 2$ in QCD. However the full non-perturbative coupling must still differ by a $\delta\alpha$ term, since the perturbative coupling is non-causal below $N_f = N_f^*$.

N_f. In this case there can be no space-like Landau singularity, and causality must be violated by the appearance of *complex* Landau singularities on the first sheet of the k^2 plane. It is natural to assume, as suggested by the 2-loop example below, that they arise as the result of the continuous migration to the first sheet, through the time-like cut, of some second sheet singularities already present when $N_f > N_f^*$. I shall assume that this is the scenario which prevails in QCD. As the simplest example, consider the two-loop coupling, which satisfies the renormalization group equation

$$\frac{d\alpha}{d\log k^2} = -\beta_0\alpha^2 - \beta_1\alpha^3 \tag{4.1}$$

If $\beta_0 > 0$ but $\beta_1 < 0$, there is an IR fixed point at $\alpha_{IR} = -\frac{\beta_0}{\beta_1}$. It has been shown in [9, 10, 6] that this coupling has a pair of complex conjugate Landau singularities on the second (or higher) sheet if

$$0 < \gamma_{2-loop} = -\frac{\beta_0^2}{\beta_1} < 1 \tag{4.2}$$

where γ_{2-loop} is the 2-loop critical exponent (see below). For $\gamma_{2-loop} > 1$, the second sheet singularities move to the first sheet through the time-like cut, which is reached when $\gamma_{2-loop} = 1$. The latter condition thus determines the bottom of the conformal window in this model. Note that in the limit $\beta_1 \to 0^-$ where $\gamma_{2-loop} = +\infty$, one gets the one loop coupling and the complex conjugate singularities collapse to a space-like Landau pole. This limit is thus the analogue of the $N_f \to -\infty$ limit in QCD.

5. An equation to determine the bottom of the conformal window in QCD

Assuming from now on that the second scenario described above applies, i.e. that there is an IR fixed point even for $N_f < N_f^*$, I shall argue that the condition

$$0 < \gamma < 1 \tag{5.1}$$

is both necessary and sufficient for causality for a broad class of beta functions, and therefore that the lower boundary of the conformal window can be obtained from the equation

$$\gamma(N_f^*) = 1 \tag{5.2}$$

where γ is the critical exponent defined as the derivative of the beta function at the fixed point

$$\gamma = \frac{d\beta(\alpha)}{d\alpha}|_{\alpha=\alpha_{IR}} \tag{5.3}$$

As is well known, the critical exponent is a *universal* quantity, independant of the definition of the coupling, and eq.(5.2) is a renormalization scheme invariant condition, as it should. The argument proceeds in two steps.

i) It was shown[3] in [6] that eq.(5.1) is a necessary condition for causality. I shall consider a restricted (and improved) version of the argument in [6], where one assumes the coupling flows to an ultraviolet (UV) Landau singularity in the strong coupling phase $\alpha > \alpha_{IR}$ (one can assume for instance a pole in the beta function at $\alpha_P > \alpha_{IR}$), i.e. no UV fixed point for $\alpha > \alpha_{IR}$. Solving the renormalization group equation

$$\frac{d\alpha}{d\log k^2} = \beta(\alpha) \tag{5.4}$$

around $\alpha = \alpha_{IR}$, one gets

$$\alpha(k^2) = \alpha_{IR} - \left(\frac{k^2}{\Lambda^2}\right)^{\gamma} + ... \tag{5.5}$$

There are thus rays

$$k^2 = |k^2| \exp\left(\pm\frac{i\pi}{\gamma}\right) \tag{5.6}$$

in the complex k^2 plane, which in the infrared limit $|k^2| \to 0$ are mapped by eq.(5.5) to positive real values of the coupling *larger* then α_{IR}. Assuming an expansion

$$\beta(\alpha) = \gamma(\alpha - \alpha_{IR}) + \gamma_1(\alpha - \alpha_{IR})^2 + ... \tag{5.7}$$

the corrections to eq.(5.5) are given by a series

$$\log(k^2/\Lambda^2) = \frac{1}{\gamma}\log(\alpha_{IR} - \alpha) + \frac{\gamma_1}{\gamma^2}(\alpha_{IR} - \alpha) + ... \tag{5.8}$$

with *real* coefficients, showing that the only contribution to the phase for $\alpha > \alpha_{IR}$ comes from the logarithm on the right hand side of eq.(5.8). The trajectories in the k^2 plane which map to the $\alpha > \alpha_{IR}$ region are thus straight lines to all orders of perturbation theory around α_{IR}. This fact suggests that even away from the infrared limit, these trajectories are given by the rays eq.(5.6). The assumed absence of an UV fixed point for $\alpha > \alpha_{IR}$ implies that the coupling will flow along these rays to an UV Landau singularity, reached at some finite value of $|k^2|$. If $\gamma > 1$ the rays, hence also the Landau singularities, are located on the first sheet of the k^2 plane, showing that eq.(5.1) is a necessary condition for causality.

2) To assert whether eq.(5.1) is also sufficient for causality, one has to make sure that *no other sources* of Landau singularities are present, but the one arising from the $\alpha > \alpha_{IR}$ region.

It is clearly impossible to discuss all possible singularities and check the last point without the knowledge of the full beta function. I shall focuss only on a potential problem arising from an eventual UV Landau singularity at $\alpha < 0$, in the domain of

[3] The general argument in [6] is not quite correct however, at the difference of the more restricted version used here.

attraction of the *trivial* IR fixed point $\alpha = 0^-$. Indeed at weak coupling the solution of the RG equation is controlled by the 2-loop beta function

$$\log(k^2/\Lambda^2) = \frac{1}{\beta_0 \alpha} + \frac{\beta_1}{\beta_0^2} \log \alpha + const + ... \tag{5.9}$$

where the *const* is real. For $\alpha < 0$ the right hand side of eq.(5.9) acquires a $\pm i\pi \frac{\beta_1}{\beta_0^2}$ imaginary part, which implies that the rays

$$k^2 = |k^2| \exp\left(\pm i\pi \frac{\beta_1}{\beta_0^2}\right) \tag{5.10}$$

map to the $\alpha < 0$ region. Along the rays eq.(5.10), we are effectively in a QED like situation: increasing $|k^2|$, the coupling is either attracted to a non-trivial UV fixed point, or reaches an UV Landau singularity at some finite $|k^2|$. In the latter case, one must require that the condition:

$$|\frac{\beta_0^2}{\beta_1}| < 1 \tag{5.11}$$

is satisfied, which will confine the Landau singularity to the second (or higher) sheet.

However in QCD, condition eq.(5.11) is *never* obtained when $\beta_1 > 0$. Therefore, if the bottom of the conformal window occurs at such low[4] N_f that $\beta_1 > 0$ (as shall be found below), one has to assume that a *non-trivial* (finite or infinite) UV fixed point α_{UV} is present[5] at *negative* α, to suppress this potential source of Landau singularity within the conformal window!

The condition eq.(5.11) coincides with the 2-loop causality condition eq.(4.2) if $\beta_1 < 0$, but it applies to a general beta function, and also when $\beta_1 > 0$. It is worth mentionning this condition is *always* violated [6]in the lower part of the conformal window in SQCD, and the previous argument thus implies the existence of a negative UV fixed point in this case too. In fact the "exact" NSVZ beta function for $N_f = 0$ does exhibit an (infinite) UV fixed point as $\alpha \to -\infty$, which might be the parent of a similar one present within Seiberg conformal window.

In summary, condition eq.(5.1) is both necessary and sufficient for causality within the class of beta functions which admit a) a negative UV fixed point in the domain of

[4]Eq.(5.11) can also be violated at larger N_f where $\beta_1 < 0$ if $|\beta_1|$ is small enough.

[5]Note that the non-trivial UV fixed point is actually not relevant to the proper analytic continuation of the coupling at complex k^2, which must be consistent with (UV) asymptotic freedom. This means that in presence of this fixed point, the correct analytic continuation must involve *complex* rather then negative values of α along the rays eq.(5.10), and one should approach the non-trivial (rather then the trivial) IR fixed point as $|k^2| \to 0$, and the trivial (rather then the non-trivial) UV fixed point as $|k^2| \to \infty$. This is possible since the solution of eq.(5.9) is not unique for a given (complex) k^2. For the same reason, any eventual IR Landau singularity arising from the region $\alpha < \alpha_{UV}$, in the domain of attraction of the non-trivial UV fixed point, is not relevant to the correct analytic continuation. On the other hand, any UV Landau singularity in the domain of attraction of either the trivial or the non-trivial IR fixed points as considered above *is relevant* to the proper analytic continuation, since the coupling will flow to the only UV fixed point available, namely the *trivial* one, once the UV Landau singularity is passed.

attraction of the trivial 0^- IR fixed point, b) a positive IR fixed point with no positive UV fixed point in its domain of attraction and c) no *complex* Landau singularities (such as a complex pole in the beta function). A minimal example satisfying these requirements is the 3-loop beta function

$$\beta(\alpha) = -\beta_0\alpha^2 - \beta_1\alpha^3 - \beta_2\alpha^4 \tag{5.12}$$

with $\beta_0 > 0$ and $\beta_2 < 0$ (β_1 can have any sign).

6. Computing N_f^* through the Banks-Zaks expansion

One can try to use the Banks-Zaks expansion to compute γ and determine N_f^*, the lower boundary of the conformal window. The Banks-Zaks expansion [3, 4, 5] is an expansion of the fixed point in powers of the distance $N_f - N_f^0$ from the top of the conformal window, where N_f^0 is the number of flavors where β_0 vanishes ($N_f^0 = 16.5$ in QCD). The solution of the equation

$$\beta(\alpha) = -\beta_0\alpha^2 - \beta_1\alpha^3 - \beta_2\alpha^4 + ... = 0 \tag{6.1}$$

in the limit $\beta_0 \to 0$, with β_i $(i \geq 1)$ finite is obtained as a power series

$$\alpha_{IR} = a + \mathcal{O}(a^2) \tag{6.2}$$

where the expansion parameter $a \equiv \frac{8}{321}(16.5 - N_f)$ is proportionnal to β_0. The Banks-Zaks expansion for the critical exponent eq.(5.3)is presently known [7] up to next-to-next-to leading order :

$$\gamma = \frac{107}{16}a^2(1 + 4.75a - 8.89a^2 + ...) \tag{6.3}$$

Using the trunkated expansion eq.(6.3), one finds that $\gamma < 1$ for $N_f \geq 5$, with $\gamma = 1$ reached for $N_f = N_f^* \simeq 4$. To assess whether it is reasonable to use perturbation theory down to $N_f = N_f^*$, let us look at the magnitude of the successive terms within the parenthesis in eq.(6.3). They are given by: $1, 1.44, -0.82$. Although the next to leading term gives a very large correction, and the series seem at best poorly converging at $N_f = N_f^*$, one can observe that the next-to next to leading term still gives a moderate correction to the sum of the first two terms, which might be considered together [6] as building the "leading" contribution, since they are both derived from information contained in the minimal 2-loop beta function necessary to get a non-trivial fixed point. Indeed, keeping only the first two terms in eq.(6.3), one finds that $\gamma = 1$ is reached for $N_f = N_f^* \simeq 6$. On the other hand, using a $[1, 1]$ Pade approximant as a model[6] for extrapolation of the perturbative series, one gets

[6]The alternative $[0, 2]$ Pade yields a result ($\gamma < 0.26$) inconsistent with the present framework. It also predicts a not very plausible $\mathcal{O}(a^5)$ coefficient of $\simeq -192$.

$$\gamma = \frac{107}{16}\ a^2\ \frac{1 + 6.62a}{1 + 1.87a} \tag{6.4}$$

which yields $\gamma = 1$ for $N_f = N_f^* \simeq 5$. The figure below shows γ as a function of N_f:

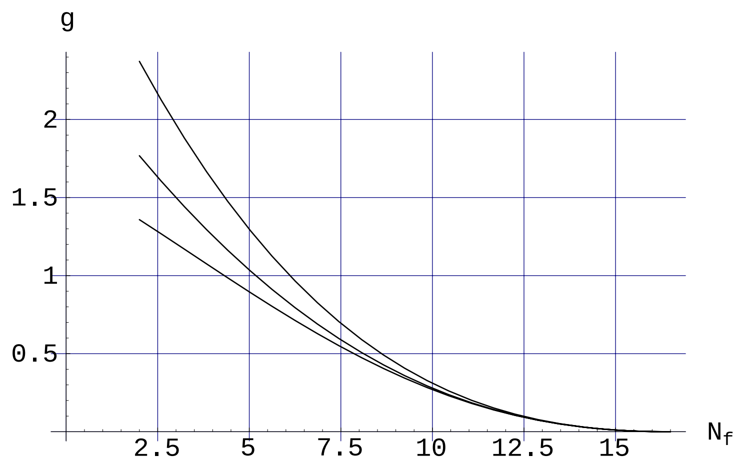

Figure 1: The critical exponent as a function of N_f: top: $\mathcal{O}(a^3)$ order; middle: Pade; bottom: $\mathcal{O}(a^4)$ order.

Note that in the obtained range of N_f^* values ($4 < N_f^* < 6$), β_1 is still positive[7] and of the same sign as β_0, so that the fixed point must arise from the contributions of higher then 2 loop beta function corrections, although I am assuming the Banks-Zaks expansion is still converging there.

7. Conclusions

1) A direct, *physical* motivation for Landau singularities is suggested, assuming a two-phase structure of QCD: they should trace the lower boundary N_f^* of the conformal window. This approach avoids the notoriously tricky disentangling of the "perturbative" from the "non-perturbative" part of the QCD amplitudes within the conformal window, since they are by definition entirely "perturbative" there. On the other hand, such a disentangling is naturally achieved below the conformal window, by introducing the analytic continuation of the conformal window amplitudes to the $N_f < N_f^*$ region.

2) Assuming that the *perturbative* QCD coupling has a non-trivial IR fixed point α_{IR} even *below* N_f^* leads to the equation $\gamma(N_f = N_f^*) = 1$ to determine N_f^* from the critical exponent γ at the fixed point. Using the available terms in the Banks-Zaks expansion, this equation yields $4 \leq N_f^* \leq 6$. Note that this condition is inconsistent with the existence of an analogue of Seiberg duality in QCD, which would rather require

[7] β_1 changes sign for $N_f \simeq 8$.

$\gamma(N_f = N_f^*) = 0$. It would clearly be desirable to have more terms to better control the accuracy of the Banks-Zaks expansion.

3) Some conditions on the QCD beta function are required: the *only* source of (UV) Landau singularities must come from the $\alpha > \alpha_{IR}$ region. Assuming the above range of values of N_f^* is correct, one needs in particular a *negative* UV fixed point in the QCD beta function. A similar negative UV fixed point is required in the SQCD case, where duality fixes the conformal window.

4) It is possible the *finite* IR fixed point persists in the perturbative QCD coupling down to the $N_f \to -\infty$ one-loop limit. A simple example is provided by a beta function with one positive pole α_P (the required Landau singularity) and two opposite sign zeroes (α_{IR} and α_{UV}):

$$\beta(\alpha) = -\beta_0 \alpha^2 \frac{(1 - \frac{\alpha}{\alpha_{IR}})(1 - \frac{\alpha}{\alpha_{UV}})}{1 - \frac{\alpha}{\alpha_P}}$$

where $\alpha_{UV} < 0$ and $0 < \alpha_{IR} < \alpha_P$. The one-loop limit is achieved for $\alpha_{IR} = \alpha_P$ and $\alpha_{UV} = -\infty$. A paper developing further these issues is under preparation.

Acknowledgments

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References

[1] See e.g. Yu.L. Dokshitzer, in *International Conference "Frontiers of Matter"*, Blois, France, June 1999 [hep-ph/9911299].

[2] G. Grunberg, *Phys.Lett.* **B349** 469 (1995).

[3] T. Banks and A. Zaks, *Nucl. Phys.* **B196** (1982) 189.

[4] A.R. White, *Phys.Rev.* **D29** (1984) 1435; *Int. J. Mod. Phys.* **A8** (1993) 4755 [hep-th/9303053].

[5] G. Grunberg, *Phys.Rev.* **D46** (1992) 2228.

[6] E. Gardi and G. Grunberg, *JHEP* **03** (1999) 024 [hep-th/9810192].

[7] P.M. Stevenson, *Phys.Lett.* **B331** (1994) 187 [hep-ph/9402276]; S.A. Caveny and P.M. Stevenson, hep-ph/9705319.

[8] E. Gardi and M. Karliner, *Nucl. Phys.* **B529** (1998) 383 [hep-ph/9802218].

[9] N.G. Uraltsev, private communication.

[10] E. Gardi, G. Grunberg and M. Karliner, *JHEP* **07** (1998) 007 [hep-ph/9806462].